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Uniqueness of the $(p, q) - 1$-Normal Matrix of Nodes in an Interval*

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It is proved that for any nonnegative integers p and q there exists a unique $(p, q) - 1$-normal matrix of nodes in a finite interval.

1. INTRODUCTION

Let

$$\mathbf{X} = [x_{nk}], \qquad x_{nk} \in [a, b],\ n = 1, 2, ..., \ k = 1, 2, ..., n, \qquad (1.1_1)$$
$$x_{nk} \neq x_{nh} \quad \text{if} \quad k \neq h,\ n \geqslant 2,$$

be a matrix of nodes x_{nk} belonging to a finite interval $[a, b]$ and let X_n represent the nth row of $\mathbf{X}$: $X_n = [x_{n1} x_{n2} \cdots x_{nn}]$.

Consider the functions v_{nk} defined by

$$v_{nk}(x) := 1 - [\omega_n''(x_{nk})/\omega_n'(x_{nk})](x - x_{nk}), \qquad n \in N^+,\ 1 \leqslant k \leqslant n,$$

where

$$\omega_n(x) := (x - x_{n1})(x - x_{n2}) \cdots (x - x_{nn}).$$

DEFINITION 1.1. The matrix $\mathbf{X}$ in (1.1_1) is said to be a normal matrix of nodes in the interval $[a, b]$ if for all $x \in [a, b]$ and for all n and k, $n \in N^+$, $1 \leqslant k \leqslant n$, the inequality $v_{nk}(x) \geqslant 0$ holds. In addition, a normal matrix $\mathbf{X}$ is said to be ρ-normal for some $\rho > 0$ if for all $x \in [a, b]$ and for all n and k, $n \in N^+$, $1 \leqslant k \leqslant n$, the inequality $v_{nk}(x) \geqslant \rho$ holds.

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Normal matrices were introduced by Fejér [4, 5], who showed their significance in both Lagrange and Hermite interpolation. Subsequently, they have been widely studied by several authors (see, e.g., [3, 7, 1, 6]). The reader is also referred to [8], where the principal results obtained by Fejér about normal matrices are summarized.

Generalizations of normality and ρ-normality have been also introduced and studied (see, e.g., [11, 10, 12–14, 9]).

Most of them can be carried over to the so-called H_{npq} process, that is to say, a Hermite–Fejér type interpolation which includes boundary conditions [14, 9]. In this case the generalization concerns the function v_{nkpq}, which replaces the function v_{nk}. For any nonnegative integers p and q and for all n and k, $n \in N^+$, $1 \leqslant k \leqslant n$, the function v_{nkpq} is defined by

$$v_{nkpq}(x) := 1 + [p/(b - x_{nk}) - q/(x_{nk} - a) - \omega_n''(x_{nk})/\omega_n'(x_{nk})](x - x_{nk}).$$

DEFINITION 1.2. Let p and q be nonnegative integers. The matrix $\mathbf{X}$ in (1.1_1) is said to be a (p, q)-normal matrix of nodes in the interval $[a, b]$ if for all $x \in [a, b]$ and for all n and k, $n \in N^+$, $1 \leqslant k \leqslant n$, the inequality $v_{nkpq}(x) \geqslant 0$ holds. In addition, a (p, q)-normal matrix $\mathbf{X}$ is said to be $(p, q) - \rho$-normal for some $\rho > 0$ if for all $x \in [a, b]$ and for all n and k, $n \in N^+$, $1 \leqslant k \leqslant n$, the inequality $v_{nkpq}(x) \geqslant \rho$ holds.

Remark 1.1. The arrangement of the entries in a same row X_n is of no interest in this paper. Thus, from now on, the matrices (1.1_1) whose rows consist of the same nodes are regarded as a unique matrix of nodes and the arrangement

$$x_{n1} < x_{n2} < \cdots < x_{nn}, \qquad n \geqslant 2, \qquad (1.1_2)$$

is assumed for the sake of simplicity.

The most popular examples of (p, q)-normal matrices in $[a, b] = [-1, 1]$ are the matrices formed by the zeros of the Jacobi polynomials. Such two-parameter matrices will be denoted in this paper by $\mathbf{X}^{[\alpha, \beta]}$, $\alpha, \beta \geqslant -1$. For any $\alpha, \beta \geqslant -1$, $\mathbf{X}^{[\alpha, \beta]}$ is the matrix defined by

$$\omega_n = p_n^{[\alpha, \beta]}, \qquad n \in N^+, n > -\alpha - \beta - 1, \qquad (1.2)$$

where $p_n^{[\alpha, \beta]}$ indicates the unique monic polynomial solution of the differential equation

$$(1 - x^2)\, y'' + [\beta - \alpha - (\alpha + \beta + 2)x]\, y' + n(n + \alpha + \beta + 1)\, y = 0. \quad (1.3)$$

The following cases deserve particular attention in this paper. One has

$$p_n^{[-1,\beta]}(x) = (x-1)\, p_{n-1}^{[1,\beta]}(x), \qquad \beta > -1, \tag{1.4$_1$}$$

$$p_n^{[\alpha,-1]}(x) = (x+1)\, p_{n-1}^{[\alpha,1]}(x), \qquad \alpha > -1, \tag{1.4$_2$}$$

$$p_n^{[-1,-1]}(x) = (x^2-1)\, p_{n-2}^{[1,1]}(x), \qquad n \geqslant 2. \tag{1.4$_3$}$$

The zeros of $p_n^{[\alpha,\beta]}$ will be denoted by $x_{nk}^{[\alpha,\beta]}$.

Remark 1.2. The first row of $\mathbf{X}^{[-1,-1]}$ is not defined by (1.2) and the differential equation in (1.3) does not have a unique monic polynomial solution when $\alpha=\beta=-1$, $n=1$. (In this case, the solutions of (1.3) are the polynomials of degree at most 1.) This has to be understood in the sense that the first row in $\mathbf{X}^{[-1,-1]}$, that is to say, the node $x_{11}^{[-1,-1]}$, has to be considered as arbitrarily preassigned. As it will be seen later (see the proof of Case (iv), Section 2), this corresponds to a precise theoretical reason. For any other choice of $\alpha, \beta \geqslant -1$, $n \in N^+$, the differential equation in (1.3) has a unique monic polynomial solution and this is of degree n.

Examples discussed in [4, 5, 8, 9] are summarized in the following

EXAMPLE. The matrix $\mathbf{X}^{[\alpha,\beta]}$ is a (p, q)-normal matrix of nodes in $[-1, 1]$ if and only if $p-1 \leqslant \alpha \leqslant p$, $q-1 \leqslant \beta \leqslant q$. In particular, $\mathbf{X}^{[\alpha,\beta]}$ is a $(p,q)-\rho$-normal matrix, $\rho = \min\{p-\alpha,\ q-\beta\}$, if $p-1 \leqslant \alpha < p$, $q-1 \leqslant \beta < q$.

The matrices $\mathbf{X}^{[-1,-1]}$ and $\mathbf{X}^{[0,0]}$ have been studied quite extensively. The former is 1-normal. It was first considered by the French astronomer R. Radau who used it in mechanical quadratures. The latter is $(1,1)-1$-normal. It was used by Egerváry and Turán [2] to construct the "most economical" interpolation process and then it was considered in the papers [11, 10] which concern the particular case $p=q=1$ [H_{n11} process and $(1,1)$-normal matrices are called quasi-Hermite–Fejér interpolation and quasi-normal matrices in these papers].

Further examples of normal matrices were first given by Freud [6]. He proved that the matrix $\mathbf{X}$ whose nth row X_n, $n \in N^+$, consists of the zeros of the polynomial $P_n^{[0]} + AP_{n-1}^{[0]}$ (where $P_m^{[0]} = p_m^{[0,0]}/p_m^{[0,0]}(1)$) is a normal matrix in $[-1, 1]$ for every $|A| \leqslant 1$. Later, Vértesi [13] characterized the shifts in the rows X_n which allow one to get an infinite number of $\rho/2-(p,q)$-normal matrices from given $\rho-(p,q)$-normal matrices. He also constructed further new examples by adding the endpoints -1 and 1 to suitable combinations of $x_{nk}^{[\alpha,\beta]}$'s.

In this short note the limit case $\rho=1$ is studied. Note that, according to the above definitions, ρ can not be greater than 1. In fact, one has $v_{nkpq}(x_{nk}) = 1$.

The Theorem in Section 2 proves the uniqueness of the $(p, q)-1$-normal matrix of nodes in an interval.

2. Uniqueness of the $(p, q)-1$-Normal Matrix

The case $[a, b]=[-1, 1]$ is assumed throughout this section for the sake of simplicity. There is no loss of generality in doing so. This can be easily seen by an argument based upon the affine mappings in the real line.

The following four cases are distinguished in the Theorem below: Case (i), $p, q \geqslant 1$; Case (ii), $p=0$, $q \geqslant 1$; Case (iii), $p \geqslant 1$, $q=0$; Case (iv), $p=q=0$.

It is implicitly contained in Definition 1.2—and in the H_{npq} process too—that the conditions

$$-1<x_{nk}<1, \qquad 1 \leqslant k \leqslant n, \tag{2.1}$$

must be true for each $n \in N^+$ in Case (i). Analogously (see also (1.1_2)), the conditions

$$-1<x_{n1}<x_{n2}< \cdots <x_{nn} \leqslant 1 \tag{2.2}$$

and

$$-1 \leqslant x_{n1}<x_{n2}< \cdots <x_{nn}<1 \tag{2.3}$$

must be satisfied for each $n \in N^+$ in Case (ii) and in Case (iii), respectively. On the contrary, no further condition on nodes has to be added to those in (1.1_2) when Case (iv) is considered:

$$-1 \leqslant x_{n1}<x_{n2}< \cdots <x_{nn} \leqslant 1. \tag{2.4}$$

Theorem. *For every nonnegative integers p and q, the matrix $\mathbf{X}^{[p-1, q-1]}$ is the unique $(p, q)-1$-normal matrix of nodes in $[-1, 1]$.*

Proof. Assume that the matrix $\mathbf{X}$ in (1.1) is $(p, q)-1$-normal in the interval $[a, b]=[-1, 1]$. From Definition 1.2 it follows that it must be

$$p/(1-x_{nk})-q/(1+x_{nk})-\omega_n''(x_{nk})/\omega_n'(x_{nk})=0$$
$$\text{whenever} \quad -1<x_{nk}<1. \tag{2.5}$$

From now on, it is convenient to treat the above listed cases separately. Nevertheless, the arguments used in the four cases are quite similar.

Case (i). $p, q \geqslant 1$. It is the simplest one to be discussed. Let $n \in N^+$. It follows from (2.1) and (2.5) that

$$(1 - x_{nk}^2)\,\omega_n''(x_{nk}) + [q - p - (p+q)\,x_{nk}]\,\omega_n'(x_{nk}) = 0, \qquad 1 \leqslant k \leqslant n.$$

Therefore, the polynomial P_n,

$$P_n(x) := (1 - x^2)\,\omega_n''(x) + [q - p - (p+q)x]\,\omega_n'(x), \tag{2.6}$$

is a polynomial of degree n which vanishes at the n points x_{nk}, $1 \leqslant k \leqslant n$. Thus,

$$P_n = c_n \omega_n, \qquad c_n \text{ constant}, \tag{2.7}$$

and it immediately follows, by comparing (2.6) and (2.7), that the constant c_n must be equal to $-n(n+p+q-1)$

$$c_n = -n(n+p+q-1). \tag{2.8}$$

From (2.6), (2.7), and (2.8) it can be seen that ω_n is a solution of the differential equation

$$(1-x^2)\,y'' + [q-p-(p+q)x]\,y' + n(n+p+q-1)\,y = 0. \tag{2.9}$$

A comparison between (1.3) and (2.9) ($\alpha = p-1$, $\beta = q-1$) shows that in Case (i) the unique monic polynomial solution of (2.9) is $p_n^{[p-1, q-1]}$ (see also Remark 1.2), and this concludes the proof.

Case (ii). $p = 0$, $q \geqslant 1$. According to the definition of $\mathbf{X}^{[-1, q-1]}$ (see (1.2) and (1.4_1)), the assertion can be proved by showing that the conditions

$$\omega_n(x) = p_n^{[-1, q-1]}(x) = (x-1)\,p_{n-1}^{[1, q-1]}(x), \qquad n \in N^+, \tag{2.10}$$

necessarily hold.

To do so, observe that, in Case (ii), (2.5) becomes

$$Q_{n-1}(x_{nk}) = 0 \qquad \text{whenever} \quad -1 < x_{nk} < 1,$$

where Q_{n-1} denotes the polynomial defined by

$$Q_{n-1}(x) := (1+x)\,\omega_n''(x) + q\omega_n'(x). \tag{2.11}$$

Since Q_{n-1} is of degree $n-1$ in Case (ii), this implies that the equality $x_{nn} = 1$ must hold in (2.2).

Thus, (2.10) is proved in the particular case of $n=1$. Furthermore, for any $n \geqslant 2$ it must be

$$\omega_n(x)=(x-1)\,\omega_{n-1}(x), \quad \omega_{n-1}(x)=(x-x_{n1})\cdots(x-x_{n,n-1}), \tag{2.12$_1$}$$

$$Q_{n-1}(x_{nk})=0, \qquad 1 \leqslant k \leqslant n-1, \tag{2.12$_2$}$$

and from (2.11) and (2.12) the following conditions easily follow

$$\begin{aligned}(x_{nk}^2-1)\,\omega_{n-1}''(x_{nk})+[2-q+(2+q)\,x_{nk}]\,\omega_{n-1}'(x_{nk})&\\ =0, \qquad 1 \leqslant k \leqslant n-1.&\end{aligned} \tag{2.13}$$

Now an argument which is perfectly analogous to that in Case (i) can be used by first considering the polynomial P_{n-1},

$$P_{n-1}(x):=(x^2-1)\,\omega_{n-1}''(x)+[2-q+(2+q)x]\,\omega_{n-1}'(x), \tag{2.14}$$

and then by deriving from (2.13) and (2.14) that ω_{n-1} has to be a solution of the differential equation

$$(1-x^2)\,y''+[q-2-(q+2)x]\,y'+(n-1)(n+q)\,y=0. \tag{2.15}$$

Finally, a comparison between (1.3) and (2.15) ($\alpha=1$, $\beta=q-1$, $n-1$ in place of n) shows that in Case (ii) the unique monic polynomial solution of (2.15) is $p_{n-1}^{[1,q-1]}$ (see also Remark 1.2), and this—taking into account (2.12$_1$)—completes the proof of (2.10).

Case (iii). $p \geqslant 1$, $q=0$. The proof can be deduced from that in Case (ii) by obvious modifications.

Case (iv). $p=q=0$. Note that one has $\omega_1''(x) \equiv 0$. This implies $v_{1100}(x) \equiv 1$ no matter what $X_1=x_{11}$ is.

Thus, *uniqueness* of the 1-normal matrix stands for *uniqueness up to the first row* X_1. Furthermore, n can be assumed greater than or equal to 2 and, according to the definition of $\mathbf{X}^{[-1,-1]}$ (see (1.2) and (1.4$_3$)), the assertion can be proved by showing that the conditions

$$\omega_n(x)=p_n^{[-1,-1]}(x)=(x^2-1)\,p_{n-2}^{[1,1]}(x), \qquad n \geqslant 2, \tag{2.16}$$

necessarily hold.

To do so, observe that, as a consequence of (2.4) and (2.5), $\omega_n''(x_{nk})$ must be equal to zero if x_{nk} belongs to the open interval $(-1, 1)$. Since ω_n'' is a polynomial of degree $n-2$, no more than $n-2$ nodes can belong to $(-1, 1)$ and the following equations must be satisfied (see also (1.1$_2$))

$$x_{n1}=-x_{nn}=-1, \tag{2.17$_1$}$$

$$\omega_n''(x_{nk})=0, \qquad k=2, 3, \ldots, n-1 \ (\text{if } n>2). \tag{2.17$_2$}$$

Conditions (2.17) prove (2.16) in the particular case of $n=2$. To complete the proof, assume $n>2$. It follows from (2.17_1)

$$\omega_n(x)=(x^2-1)\,\omega_{n-2}(x), \quad \omega_{n-2}(x)=(x-x_{n2})\cdots(x-x_{n,n-1}). \tag{2.18}$$

Now, the following $n-2$ equalities can be obtained from (2.17_2) and (2.18)

$$(x_{nk}^2-1)\,\omega''_{n-2}(x_{nk})+4x_{nk}\,\omega'_{n-2}(x_{nk})=0, \qquad k=2, 3, ..., n-1, \tag{2.19}$$

and then an argument analogous to those used to conclude the proof in the other cases can be invoked by first considering the polynomial P_{n-2},

$$P_{n-2}(x):=(x^2-1)\,\omega''_{n-2}(x)+4x\omega'_{n-2}(x), \tag{2.20}$$

and afterwards by deriving from (2.19) and (2.20) that ω_{n-2} has to be a solution of the differential equation

$$(1-x^2)\,y''-4xy'+(n-2)(n+1)\,y=0. \tag{2.21}$$

Finally, a comparison between (1.3) and (2.21) ($\alpha=\beta=1$, $n-2$ in place of n) shows that in Case (iv) the unique monic polynomial solution of (2.21) is $p_{n-2}^{[1,1]}$ (see also Remark 1.2), and this—taking into account (2.18)—completes the proof of (2.16).

Remark. For the sake of completeness, it has to be mentioned that a definition of normal matrix which is even more general than that in Definition 1.2 is presented by Vértesi in [12].

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REFERENCES

1. D. L. BERMAN, On the interpolation process of Hermite, *Dokl. Akad. Nauk SSSR (N.S.)* **58** (1947), 1569–1571. [In Russian]
2. E. EGERVÁRY AND P. TURÁN, Notes on interpolation, V, *Acta Math. Acad. Sci. Hungar.* **9** (1958), 259–267.
3. P. ERDŐS AND P. TURÁN, On interpolation, II, *Ann. of Math.* **39** (1938), 703–724.
4. L. FEJÉR, Lagrangesche Interpolation und die zugehorigen konjugierten Punkte, *Math. Ann.* **106** (1932), 1–55.
5. L. FEJÉR, On the characterization of some remarkable systems of points of interpolation by means of conjugate points, *Amer. Math. Monthly* **41** (1934), 1–14.

6. G. FREUD, On Hermite–Fejér interpolation process, *Studia Sci. Math. Hungar.* **7** (1972), 307–316.
7. G. GRÜNWALD, On the theory of interpolation, *Acta Math.* **75** (1943), 219–245.
8. I. P. NATANSON, "Constructive Function Theory," Vol. III, Ungar, New York, 1965.
9. P. RABINOWITZ AND P. VÉRTESI, Hermite–Fejér interpolation with boundary conditions for ρ-normal sets, *J. Approx. Theory* **63** (1990), 98–107.
10. J. SÁNTA, Convergence theorems of quasi-Hermite–Fejér interpolation, *Publ. Math. Debrecen* **22** (1975), 23–29.
11. P. SZÀSZ, On quasi-Hermite–Fejér interpolation, *Acta Math. Acad. Sci. Hungar.* **10** (1959), 413–439.
12. P. VÉRTESI, Hermite–Fejér type interpolations, III, *Acta Math. Acad. Sci. Hungar.* **34** (1979), 67–84.
13. P. VÉRTESI, ρ-Normal point systems, *Acta Math. Acad. Sci. Hungar.* **34** (1979), 267–277.
14. P. VÉRTESI, Hermite–Fejér type interpolations. IV. Convergence criteria for Jacobi abscissas, *Acta Math. Acad. Sci. Hungar.* **39** (1982), 83–93.